Doppler Shift Target Localization

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***Abstract*—This paper outlines the problem of doppler-based target position and velocity estimation using a sensor network. The minimum number of doppler shift measurements at distinct generic sensor positions to have a finite number of solutions, and later, a unique solution for the unknown target position and velocity is stated analytically. Furthermore, we study the same problem where not only doppler shift measurements are collected, but also other types of measurements are available, e.g. bearing or distance to the target from each of the sensors. Later, we study the Cramer-Rao inequality associated with the doppler-shift measurements to a target in a sensor network, and use the Cramer-Rao bound to illustrate some results on optimal placements of the sensors when the goal is to estimate the velocity of the target. Some simulation results are presented in the end.**

***Index Terms*—Doppler Measurements, Location Estimation, Doppler Localization, Cramer-Rao Inequality, Fisher Information Matrix**

I. Introduction

Using doppler-shifts for position and velocity estimation has a long history; see e.g. [1]–[8]. Recently, the doppler effect has gained a renewed interest and it has been implemented for cooperative positioning in vehicular networks [9].

In this paper, we consider a scenario with n nodes with both transmitting and sensing capabilities, that are called sensors for the rest of this paper. The target has an unknown position and velocity $\mathbf{x} = [\mathbf{p}^\top \ \mathbf{v}^\top]^\top \in \mathbb{R}^4$. The position of the non-collocated sensors is given by $\mathbf{s}_i = [s_{i,1} \ s_{i,2}]^\top \in \mathbb{R}^2$, $\forall i \in \{1, \ldots, n\}$.

The measured doppler-shift is $\hat{\delta}_i$ at the i^{th} sensor and is caused by a target reflection due to a signal generated earlier by the same sensor. This frequency shift can be approximated by [10]

$$\hat{\delta}_i = \delta_i + w_i \tag{1a}$$

$$= 2\frac{f_{c,i}}{c}\left(\frac{(\mathbf{p}-\mathbf{s}_i)^\top}{\|\mathbf{p}-\mathbf{s}_i\|}\right)\mathbf{v} + w_i \tag{1b}$$

where c is the speed of light (or signal propagation) and $\|\cdot\|$ is the standard Euclidean vector norm and $f_{c,i}$ is the carrier frequency employed by this sensor. Finally, w_i is the noise variable. Note here that the localization is to be achieved instantaneously; we are not envisaging collecting information from agents at a number of successive instants of time and using them to infer position at a single instant of time (The connection with filtering methods is explored further below). There are many studies in the literature that try to solve a similar problem via collecting measurements over a time interval and feeding them into an estimator. For example, in [5] the problem of localization of a single aircraft using doppler measurements is studied. Other similar approaches can be found in [4], [6]–[8]. The analysis carried out in this paper along with the optimization method proposed can be considered as constituting a batch processing method for instantaneous estimation of the target location and velocity. This in turn might be used to initialize and improve the updates of any other implemented filter which tracks the target position as the target moves in the environment. For example, Kalman-based filters are prone to errors when the target's motion model deviates significantly from the actual target motion. Our analysis can guard against such behavior and re-initialize the filter, e.g. see [11] and references therein.

The first main contribution of this paper is that the minimum number of doppler shift measurements required to have a finite number of solutions for the unknown $\mathbf{x}$ is algebraically derived. In some scenarios, a separate piece of knowledge about the target will allow disambiguation. Later, the result is extended to the case where having a unique solution is required. Moreover, the scenarios where different types of measurements, e.g. direction-of-arrival, or distance, are available in addition to doppler shift measurements are considered. The aforementioned conclusions assume zero measurement noise; following on from this, an optimization method based on polynomial optimization methods is introduced to calculate the velocity and the position of a target where noisy doppler shift measurements are available. Later, we calculate the Cramer-Rao inequality for different sensor network configurations and present some discussions of the Fisher Information matrix, which in turn leads to the introduction of optimal sensor placement in the scenarios where a network of sensors gather doppler-shift measurements in the signal from a target.

The remaining sections of this paper are organized as follows. In the next section the main problem of interest in considered. In Section III the case where different types of measurement in addition to doppler shift measurement are available to the sensors is considered. A method based on polynomial optimization to estimate the position and the velocity of the target where doppler shift measurements are contaminated by noise is presented in Section IV. The Cramer-

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Rao inequality for the scenario considered here is calculated in Section V. In Section VI some insights into the Fisher Information matrix for the velocity estimation are presented. An illustrative example presenting the performance of the proposed optimization method is given in Section VII. Concluding remarks and future directions come in Section VIII.

II. REQUIRED MINIMUM NUMBER OF DOPPLER SHIFT MEASUREMENTS

Initially, it is assumed that the measurements are noiseless, i.e. $w_i = 0$.

$$\begin{aligned}\hat{\delta}_i &= \delta_i \\ &= 2\frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i)}{\|\mathbf{p}-\mathbf{s}_i\|}\frac{f_{c,i}}{c}.\end{aligned} \tag{2}$$

Normalizing so that $f_i \triangleq \frac{c\delta_i}{2f_{c,i}}$, we obtain

$$f_i = \frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i)}{\|\mathbf{p}-\mathbf{s}_i\|} \tag{3}$$

Now we are ready to pose the problem of interest in this section.

Problem 1. *Consider n stationary sensors at $\mathbf{s}_i \in \mathbb{R}^2$ capable of collecting noiseless doppler shift measurements from a target at position $\mathbf{p}$ moving with a nonzero velocity $\mathbf{v}$ of the form* (2).

1) *What is the minimum value for n such that there is a finite number of solutions for $\mathbf{x}$?*
2) *What is the minimum value for n such that there is a unique solution for $\mathbf{x}$?*

We limit our analysis to the case where the nodes and the target are in $\mathbb{R}^2$, however, note that the analysis for the case that they are in $\mathbb{R}^3$ is much the same. First we have the following remark.

Remark 1. *Throughout this paper, when it is stated that a property is held for generic positions of the sensors, it is meant that such a property holds for all positions of the sensors except for sensor positions in a set of measure zero.*

The answer to the first question posed in Problem 1 is formally presented in the following proposition for the case where $n = 4$ doppler shift measurements are available. This proposition states that with $n = 4$ measurements generically [1] we have a finite number of solutions and one might be able to disambiguate the solutions using other measurements, e.g. range, or bearing measurements. Moreover, disambiguation may also be made possible due to prior measurements, or to a priori knowledge about the geographic constraints on targets in the area of interest. This particularly is important for the cases where the possible solutions are widely separated, e.g., see Fig. 1. Hence this situation is of potential practical interest.

[1]There will be special positions of sensors for which $\mathbf{x}$ cannot be determined–for example if they are all collocated. However, the set of such exceptional positions is a set of measure zero; the word 'generically' captures this notion.

Proposition 1. *For $n = 4$ doppler measurements as described by* (2) *and generic positions of the sensors there is a finite number of solutions for the unknown $\mathbf{x}$.*

Proof: Denote the noiseless mapping from the agent position and velocity, $\mathbf{x} = [\mathbf{p}^\top \ \mathbf{v}^\top]^\top$ (a vector in $\mathbb{R}^4$) to measurements (another vector in $\mathbb{R}^n$) by F, where n is the number of sensors. More specifically $F(\mathbf{x}) = [f_1 \ f_2 \ f_3 \ f_4]^\top$. Let ∇F denote the Jacobian of F:

$$\nabla F = \begin{bmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial p_2} & \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_4}{\partial p_1} & \frac{\partial f_4}{\partial p_2} & \frac{\partial f_4}{\partial v_1} & \frac{\partial f_4}{\partial v_2} \end{bmatrix} \tag{4}$$

where

$$\frac{\partial f_i}{\partial p_1} = \frac{-(p_2 - s_{i,2})(s_{i,2}v_1 - s_{i,1}v_2 + p_1v_2 - p_2v_1)}{\|\mathbf{s}_i - \mathbf{p}\|^3} \tag{5}$$

$$\frac{\partial f_i}{\partial p_2} = \frac{(p_1 - s_{i,1})(s_{i,2}v_1 - s_{i,1}v_2 + p_1v_2 - p_2v_1)}{\|\mathbf{s}_i - \mathbf{p}\|^3} \tag{6}$$

$$\frac{\partial f_i}{\partial v_1} = \frac{(p_1 - s_{i,1})}{\|\mathbf{s}_i - \mathbf{p}\|} \tag{7}$$

$$\frac{\partial f_i}{\partial v_2} = \frac{(p_2 - s_{i,2})}{\|\mathbf{s}_i - \mathbf{p}\|} \tag{8}$$

It is easy to check that for generic values of $\mathbf{x}$, $\mathbf{s}_i$, $i = 1, \cdots, 4$, ∇F is not singular. Moreover, we know that the set of solutions to (2) form an algebraic variety , [12], i.e. the solution set can be defined by solving a set of multivariable polynomial equations. The reason for this is that while (2) is not a polynomial equation, it is easy to see that its zeros are also the zeros of the following polynomial equation.

$$f_i^2\|\mathbf{p}-\mathbf{s}_i\|^2 - (\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i))^2 = 0 \tag{9}$$

The set of the solutions to the equations described by (9) is known to have at least one member; that is the solution corresponding to the physical setup. The nonsingularity of the Jacobian implies that for generic values for measurements we do not have a continuous set of solutions. As a result, the variety is a zero-dmensional variety [12]. Further, any zero-dimensional variety has a finite number of points [12], and so the solution set as a subset of a zero-dimensional variety also has a finite number of points, i.e., there is a finite number of solutions for the localization problem using doppler measurements where $n \geq 4$. ■

Now we briefly consider the case where the Jacobian matrix ∇F is singular. This corresponds to those sensor and target geometries where there is an infinite number of solutions for the unknown $\mathbf{p}$ and $\mathbf{v}$. We call these geometries as *degenerate geometries*. The following proposition characterizes two of these degenerate geometries.

Proposition 2. *The Jacobian matrix ∇F given in* (4) *is singular in the following situations:*

1) *If any pair of the sensors $\mathbf{s}_1, \cdots, \mathbf{s}_4$ and the target $\mathbf{p}$ are collinear.*

2) *If* $\mathbf{v} = 0$.

Proof: To prove the first statement of this proposition it is enough to evaluate (4) for the case where two of the sensors $\mathbf{s}_1, \cdots, \mathbf{s}_4$, and $\mathbf{p}$ lie on the same line. The calculations are trivial and are omitted for brevity. For establishing the validity of the second statement it is enough to observe that the first two columns of (4) are zero when $\mathbf{v} = 0$. ∎

It is worthwhile to note that in practice any geometry in which the sensors and the target are almost collinear will also be problematic due to the presence of noise in the measurements.

After establishing that there is a finite number of solutions for generic positions of the sensors where four doppler shift measurements are available, we present an answer to the second question posed in Problem 1. Before we formally propose the answer, note that the number of unknowns is four, and that with four pieces of data we have four polynomials equations, which in general have a finite set multiple solutions (though they may not all be real). However, with five pieces of data, one expects that the associated equations generically have a unique solution. This solution is the one solution common to two selections of four. In the next proposition we prove that the position and the velocity of the target can be uniquely calculated if there are five doppler measurement available.

Proposition 3. *For* $n \geq 5$ *doppler measurements as described by* (2) *and generic positions for the sensors there is a unique solution for the unknown* $\mathbf{x}$.

Proof: From Proposition 1 we know that for $n = 4$ there is a finite number of solutions for $\mathbf{x}$, m say. Call the solutions for the position and the velocity of the target $\mathbf{x}_1, \cdots, \mathbf{x}_m$. Now, temporarily regard the position of sensor 5, $\mathbf{s}_5 = [s_{5,1}\ s_{5,2}]$ as an unknown. Consider the relationship between each of these solutions, $\mathbf{x}_i = [p_{i,1}\ p_{i,2}\ v_{i,1}\ v_{i,2}]^\top$, and the position of sensor 5, $\mathbf{s}_5$:

$$\begin{aligned} f_5^2\left((p_{i,1}-s_{5,1})^2+(p_{i,2}-s_{5,2})^2\right) - \\ (v_{i,1}(p_{i,1}-s_{5,1})+v_{i,2}(p_{i,2}-s_{5,2}))^2 = 0 \end{aligned} \tag{10}$$

This equation (regarded as an equation for the two temporarily unknown coordinates $s_{5,1}$ and $s_{5,2}$, results in two straight lines intersecting at $\mathbf{p}_i$; the true $\mathbf{s}_5$ must lie on one of these straight line pairs, namely that associated with the correct target position. We claim that generically only one such equation can be satisfied by the true $\mathbf{s}_5$. To establish the claim, we argue by contradiction. Assume that $\mathbf{x}_j$ and $\mathbf{x}_k$ define two different loci on both of which $\mathbf{s}_5$ lies. These loci at most intersect at four points. So for all positions of $\mathbf{s}_5$ except these four points, $\mathbf{x}_j$ and $\mathbf{x}_k$ cannot be simultaneously the solutions to the localization problem via doppler measurements. With similar arguments one can eliminate any multiplicity of solutions for generic values of doppler shift measurements for $n \geq 5$. ∎

The proof of Proposition 3 indicates that, given four sensors in generic positions there are isolated positions where placing a fifth sensor does not lead to a unique solution for $\mathbf{x}$. An example of this scenario is depicted in Fig. 1, where two possible straight line pairs are shown, being determined by two of the possible finite target positions computed using the measurements from sensors 1 to 4. Moreover, four positions are identified such that the placement of a fifth sensor at any of these positions will not resolve the ambiguity in the target position.

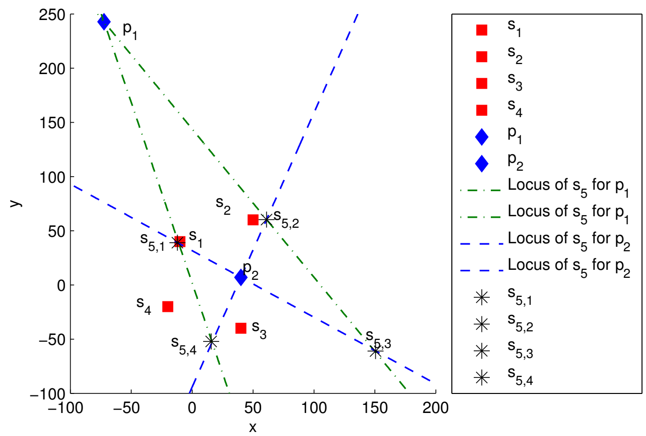

Fig. 1. Placing the fifth sensor at any of the positions indicated by * results in an ambiguous solution for the position of the target.

So far, we have considered the case where each sensor i emits a signal with a known carrier frequency $f_{c,i}$. In what comes next we consider two other cases. First, we consider a scenario that occurs commonly. In this scenario an illuminating emitter and a receiver are colocated and all the other sensors only receive the reflection of the signal emitted by the first sensor off the target. Suppose the illuminating radar and the first receiver are colocated at $\mathbf{s}_1$ and the remaining sensors are located at $\mathbf{s}_2, \ldots, \mathbf{s}_\ell$. Let f_c be the frequency of the illuminating radar. Then for $i = 1$ there holds

$$\delta_1 = 2\frac{f_c}{c}\frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_1)}{\|\mathbf{p}-\mathbf{s}_1\|}, \tag{11}$$

and for $i = 2, \ldots, \ell$, [10]

$$\delta_i = \frac{f_c}{c}\left[\frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_1)}{\|\mathbf{p}-\mathbf{s}_1\|}+\frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i)}{\|\mathbf{p}-\mathbf{s}_i\|}\right]. \tag{12}$$

We note that from this data, it is trivial to compute the set of values

$$\bar{f}_i = \frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i)}{\|\mathbf{p}-\mathbf{s}_i\|}$$

for $i = 1, 2, \ldots, \ell$, where

$$\bar{f}_i \triangleq \frac{c(2\delta_i-\delta_1)}{2f_c} \quad i = 1, 2, \ldots, \ell.$$

In other words, from a purely mathematical point of view, this scenario can be reduced to the original one, where all the $f_{c,i}$ assume the same value. Hence, the same analysis applies to this scenario.

Second, we consider the case where the target emits a signal with the frequency f_c and the sensors just measure this frequency without emitting any signal of their own. This scenario is usually known as a passive doppler localization scenario. In this case the noiseless measurements are of the form

$$\delta_i = \frac{f_c}{c}\frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i)}{\|\mathbf{p}-\mathbf{s}_i\|} \tag{13}$$

The theory applying when f_c is independently known a priori is effectively the same as the one stated previously. However, if f_c is unknown, the question arises as to whether it can be determined. For this scenario, we show that there is no way from a single set of instantaneous doppler only measurements that one can separate f_c and $\mathbf{v}$. We conclude this section by formalizing this fact and presenting a result on the case where the carrier frequency is unknown.

Proposition 4. *For $n \geq 5$ doppler measurements as described by* (13) *and generic positions for the sensors, and unknown carrier frequency f_c, there is a unique solution for the unknown $\mathbf{p}$ and for the vector $f_c\mathbf{v}$.*

Proof: Define $\nu = [\nu_1\ \nu_2]^\top \triangleq f_c\mathbf{v}$. For (13) we have

$$\delta_i = \frac{\nu^\top(\mathbf{p}-\mathbf{s}_i)}{c\|\mathbf{p}-\mathbf{s}_i\|} \tag{14}$$

From Proposition 3 we know that using five equations of the form (14), $\mathbf{p}$ and ν can be calculated uniquely. It follows easily that if one does neither know f_c nor $\mathbf{v}$ and only estimates $\nu \triangleq f_c\mathbf{v}$ then for any chosen f_c (or $\mathbf{v}$) there is a subsequent value of $\mathbf{v}$ (or f_c) that satisfies $\nu = f_c\mathbf{v}$. Thus, f_c and $\mathbf{v}$ cannot be calculated separately. ■

Remark 2. *By using the calculated value $\mathbf{p}$ in consecutive time steps, one can estimate the velocity of the agent. Using this estimated velocity and knowing ν, one can further estimate f_c.*

III. Required Minimum Number of Hybrid Measurements

In this section we study the effect of having other types of measurements additional to doppler shift measurements on calculating the velocity and the position of the target.

Before continuing further, we note that to have a *unique* solution for a set of polynomial equations there is usually a need to have *more* equations than unknowns, save in cases where the equations are linear. However, having more equations than unknowns does not of itself guarantees the existence of a unique solution; the extra equation must in some way be independent, and it may have this property in almost all circumstances, i.e. generically, even if not always. In this section we establish the cases where it can be mathematically shown that a unique solution for $\mathbf{x}$ exists when a combination of doppler and other measurements is available. We have the following results:

Proposition 5. *The following statements are true.*

1) *First, consider the case where in addition to the doppler shift measurements described earlier, the distances between each of the sensors and the target can be measured as well. Denote the distance between the sensor i and the target as d_i where*

$$d_i = \|\mathbf{p}-\mathbf{s}_i\|, \quad i = 1, \cdots, m_d \tag{15}$$

   a) *For $n \geq 2$ doppler measurements as described by* (2) *and $m_d \geq 3$ distance measurements* (15) *and generic positions of the sensors there is a unique solution for the unknown $\mathbf{x}$.*
   b) *For $n \geq 3$ doppler measurements as described by* (2) *and $m_d \geq 2$ distance measurements* (15) *and generic positions of the sensors there is a unique solution for the unknown $\mathbf{x}$.*

2) *Second, consider the case that instead of distance measurements, bearing measurements to the target at each sensor i are available, viz.*

$$\psi_i = [\psi_{i,1}\ \ \psi_{i,2}]^\top = \frac{\mathbf{p}-\mathbf{s}_i}{\|\mathbf{p}-\mathbf{s}_i\|}, \quad i = 1, \cdots, m_b \tag{16}$$

   is known at each sensor i.

   a) *For $n \geq 2$ doppler measurements as described by* (2) *and at $m_b \geq 2$ bearing measurements* (16) *and generic positions of the sensors there is a unique solution for the unknown $\mathbf{x}$.*
   b) *For $n \geq 4$ doppler measurements as described by* (2) *and only $m_b = 1$ measured bearing to the target and generic positions of the sensors there is a unique solution for the unknown $\mathbf{x}$.*

3) *Third, for $n \geq 3$ doppler measurements as described by* (2)*, at least $m_d = 1$ distance measurement* (15)*, at least $m_b = 1$ measured bearing to the target* (16)*, and generic positions of the sensors there is a unique solution for the unknown $\mathbf{x}$.*

Proof: The proof of 1a is trivial.

To prove 1b observe that the two distance measurements pin down the target to a binary ambiguity; i.e. two generic circles have two points of intersection. Using each of these positions, then from the doppler equations we obtain two sets of three linear equations in the velocity of the target. Generically, only one of these sets forms a consistent system of linear equations.

The proof of 2a is obvious.

To prove 2b without loss of generality assume that only sensor 1 can collect a bearing measurements to the target in addition to the doppler shift measurement. The rest of the sensors can only measure the doppler shift. For the doppler measurement at 1 we have

$$\begin{aligned}\delta_1 &= 2\mathbf{v}^\top\psi_1\frac{f_{c,i}}{c}\\ \delta_1 &= 2(v_1\psi_{1,1}+v_2\psi_{1,2})\frac{f_{c,i}}{c}\end{aligned} \tag{17}$$

that is a linear equation in $\mathbf{v}$. Moreover, we have

$$p_2 = s_{1,2} + \psi_{1,2}\frac{p_1 - s_{1,1}}{\psi_{1,1}} \tag{18}$$

Calculating v_2 in terms of v_1 from (17) and p_2 in terms of p_1

TABLE I
MINIMUM NUMBER OF MEASUREMENTS IN DIFFERENT SCENARIOS FOR UNIQUE TARGET POSITION AND VELOCITY LOCALIZATION

No. of Doppler Measurements	5	2	3	2	4	3
No. of Distance Measurements	0	3	2	0	0	1
No. of Bearing Measurements	0	0	0	2	1	1

from (18) and replacing them in

$$\delta_i = 2\mathbf{v}^\top \frac{\mathbf{p}-\mathbf{s}_i}{\|\mathbf{p}-\mathbf{s}_i\|}\frac{f_{c,i}}{c} \quad i = 2,3,4 \tag{19}$$

we obtain three quadratic equations in v_1 and p_1 only. Furthermore, it is known that generically three quadratic equations in two variables have a unique solution. Hence, $\mathbf{x}$ can be determined uniquely.

The proof of the last statement is very similar to that of 1b and is omitted. ■

Table I summarises the various cases we have considered. In the next section, we introduce an algorithm to estimate the position and the velocity of the target using the doppler-shift measurements measured at each of the sensor, where noise may contaminate the measurements.

IV. AN ALGORITHM TO ESTIMATE THE POSITION AND THE VELOCITY OF THE TARGET

To calculate the position and the velocity of the target in the noiseless case it is enough to solve the following system of equations.

$$f_i - \frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i)}{\|\mathbf{p}-\mathbf{s}_i\|} = 0 \quad i = 1,\dots,n \tag{20}$$

where $f_i = \dfrac{c\delta_i}{2f_{c,i}}$. Equivalently due to the fact that the target and the sensors are not collocated (i.e. $||\mathbf{p}-\mathbf{s}_i||$ is nonzero), instead of solving (20) one can solve:

$$f_i\|\mathbf{p}-\mathbf{s}_i\| - \mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i) = 0 \quad i = 1,\dots,n, \tag{21}$$

or

$$f_i\sqrt{(\mathbf{p}-\mathbf{s}_i)^\top(\mathbf{p}-\mathbf{s}_i)} - \mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i) = 0 \quad i = 1,\dots,n. \tag{22}$$

Having the square-root in (22) makes it undesirable for solving numerically. Hence, instead we consider the following set of equations.

$$f_i^2(\mathbf{p}-\mathbf{s}_i)^\top(\mathbf{p}-\mathbf{s}_i) - (\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i))^2 = 0 \quad i = 1,\dots,n. \tag{23}$$

Note that any solution of (22) is a solution of (23) but not vice versa. Assume $\mathbf{x}^\star = [\mathbf{p}^{\star\top} \ \ \mathbf{v}^{\star\top}]^\top$ is a solution to (22); then both $\mathbf{x}_1^\star = [\mathbf{p}^{\star\top} \ \ \mathbf{v}^{\star\top}]^\top$ and $\mathbf{x}_2^\star = [\mathbf{p}^{\star\top} \ \ -\mathbf{v}^{\star\top}]^\top$ satisfy (23). From Proposition 3 we know that for $n \geq 5$ (22) has a unique solution. Then it easily follows that for $n \geq 5$, (23) has exactly two solutions. Replacing $\mathbf{p}$ by $\mathbf{p}^\star$ in (20) results in a set of linear equations in $\mathbf{v}$ where only one of the values of $\mathbf{v}^\star$ or $-\mathbf{v}^\star$ satisfies it. The aforementioned analysis of the solutions for $\mathbf{x}$ in the noiseless case form the basis of the algorithm proposed in this section.

Now we consider the case where the doppler shift measurement is corrupted by noise. That is, each sensor measures $\hat{\delta}_i = \delta_i + w_i$ where w_i corresponds to the noise in the measurement carried out by sensor i. Setting $\hat{f}_i = \dfrac{c\hat{\delta}_i}{2f_{c,i}}$, we have

$$\hat{f}_i = \frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i)}{\|\mathbf{p}-\mathbf{s}_i\|}. \tag{24}$$

Note that $\hat{f}_i = f_i + \omega_i$, where $\omega_i = \dfrac{cw_i}{2f_{c,i}}$. In the noisy case neither (22) nor (23) have a solution for $\mathbf{x}$. Instead we propose solving the following minimization problem to calculate the position and the velocity of the target. Solution of similar minimization problems when range, range-difference, and bearing measurments are available are studied in [13], [14].

$$[\mathbf{p}^\star, \mathbf{v}^\star] = \underset{\mathbf{p},\mathbf{v}}{\operatorname{argmin}} \ \ J(\mathbf{p},\mathbf{v}), \tag{25}$$

where $J(\mathbf{p},\mathbf{v}) = \sum_{i=1}^{n}\left(\hat{f}_i^2\|\mathbf{p}-\mathbf{s}_i\|^2 - (\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i))^2\right)^2$. The advantage of having such a cost function is that it is a polynomial in the unknowns, and can be minimized using modern polynomial optimization methods, e.g. see [15], [16]. By solving this minimization problem we obtain two values for the position and the velocity of the target, viz. $(\mathbf{p}_1^\star, \mathbf{p}_2^\star)$ and $(\mathbf{v}_1^\star, \mathbf{v}_2^\star)$, where $\mathbf{p}_1^\star = \mathbf{p}_2^\star$ and $\mathbf{v}_1^\star = -\mathbf{v}_2^\star$. To find the correct value for the velocity as before we replace $\mathbf{p}$ by $\mathbf{p}^\star \triangleq \mathbf{p}_1^\star = \mathbf{p}_2^\star$ in (24) to obtain a set of linear equations in $\mathbf{v}$:

$$\hat{f}_i = \frac{\mathbf{v}^\top(\mathbf{p}^\star-\mathbf{s}_i)}{\|\mathbf{p}^\star-\mathbf{s}_i\|}. \tag{26}$$

It is easy to check that the linear system of equations (26) is over-determined and inconsistent, and cannot be solved. However, a least-square solution to this syetm of linear equations can be obtained. The least squares solution to this system of equations is the estimated value for the target velocity, $\mathbf{v}^\star$. This procedure is outlined in Algorithm 1.

Algorithm 1 Target Position and Velocity Estimation Using n doppler-shift Measurements Collecting at Nodes $(1,\dots,n)$ in a Sensor Network Via Polynomial Optimization.
Input: $\hat{f}_1,\dots,\hat{f}_n$
Output: $\mathbf{p}^\star$, $\mathbf{v}^\star$

Require: At least 5 sensors at generic positions.
$J(\mathbf{p},\mathbf{v}) \leftarrow \sum_{i=1}^{n}\left(\hat{f}_i^2\|\mathbf{p}-\mathbf{s}_i\|^2 - (\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i))^2\right)^2$
$[\mathbf{p}^\star,\ \mathbf{v}^\star] \leftarrow \underset{\mathbf{p},\mathbf{v}}{\operatorname{argmin}} \ \ J(\mathbf{p},\mathbf{v})$
for $i = 1 : n$ **do**
  $\hat{\psi}_i \leftarrow \dfrac{\mathbf{p}^\star - \mathbf{s}_i}{\|\mathbf{p}^\star-\mathbf{s}_i\|}$
end for
$\hat{A} \leftarrow [\hat{\psi}_1,\dots,\hat{\psi}_n]^\top$
$\hat{\mathbf{b}} \leftarrow [\hat{f}_1,\dots,\hat{f}_n]^\top$
$\mathbf{v}^\star \leftarrow \hat{A}^\dagger\hat{\mathbf{b}}$ {$\hat{A}^\dagger$ is the pseudo-inverse of $\hat{A}$.}
return $\mathbf{p}^\star$ and $\mathbf{v}^\star$

We conclude this section by briefly considering the maximum likelihood problem of

$$[\mathbf{p}^\star_{ML}, \mathbf{v}^\star_{ML}] = \underset{\mathbf{p},\mathbf{v}}{\operatorname{argmin}} \quad J_{ML}(\mathbf{p}, \mathbf{v}), \tag{27}$$

where $J_{ML}(\mathbf{p}, \mathbf{v}) = \sum_{i=1}^{n} \left(\hat{f}_i - \frac{\mathbf{v}^\top(\mathbf{p} - \mathbf{s}_i)}{\|\mathbf{p} - \mathbf{s}_i\|} \right)^2$. Solving (27) involves solving a nonlinear least squares problem that its accuracy depends on the initial guess for its solution. We note that, the output of Algorithm 1 can be used to initialize the nonlinear solver that minimizes (27). We demonstrate this later in the paper.

After establishing the minimum number of doppler measurements necessary to achieve localization, in the following sections we study the Cramer-Rao inequality for calculating the target position and velocity. Later, we propose an optimal sensor placement where the objective is to estimate the velocity of the target.

V. The Cramer-Rao Inequality

If $\mathcal{I}(\mathbf{x})$ is the Fisher information matrix, that will be formally defined below, then the Cramer-Rao inequality lower bounds the variance achievable by an unbiased estimator. For an *unbiased* estimate $\widehat{\mathbf{x}}$ of $\mathbf{x}$ we find

$$\mathrm{E}\left[(\widehat{\mathbf{x}} - \mathbf{x})(\widehat{\mathbf{x}} - \mathbf{x})^\top\right] \geq \mathcal{I}(\mathbf{x})^{-1} \tag{28}$$

If $\mathcal{I}(\mathbf{x})$ is singular then (in general) no unbiased estimator for $\mathbf{x}$ exists with a finite variance [17]. If (28) holds with equality, for some unbiased estimate $\widehat{\mathbf{x}}$, then the estimator is called *efficient* and the parameter estimate $\widehat{\mathbf{x}}$ is unique [17]. However, even if $\mathcal{I}(\mathbf{x})$ is non-singular then it is not practically guaranteed that an unbiased estimator can be recognized. Alternatively, if an unbiased estimator can be realized, it is not guaranteed that an efficient estimator exists [18].

The condition (28) says nothing about the performance and realizability of biased estimators. That is, in order to use (28) we must consider only unbiased estimators [17]. The $(i,j)^{th}$ element of $\mathcal{I}(\mathbf{x})$ is given by [19]

$$\mathcal{I}_{i,j}(\mathbf{x}) = \mathrm{E}\left[\frac{\partial}{\partial x_i} \ln\left(f_{\widehat{\mathbf{p}}}(\widehat{\mathbf{p}}; \mathbf{x})\right) \frac{\partial}{\partial x_j} \ln\left(f_{\widehat{\mathbf{p}}}(\widehat{\mathbf{p}}; \mathbf{x})\right)\right] \tag{29}$$

where $[\mathbf{p}^\top \ \mathbf{v}^\top]^\top = [x_1 \ \ldots \ x_4] \in \mathbb{R}^4$ and $f_{\mathbf{f}}(\mathbf{f}; \mathbf{x})$ is the Gaussian likelihood function. We then easily find $\mathcal{I}(\mathbf{x}) = \nabla F^\top \mathbf{R}_{\mathbf{f}}^{-1} \nabla F$. The Fisher information metric characterizes the nature of the likelihood function. If the likelihood function is sharply peaked then the true parameter value is easier to estimate from the measurements than if the likelihood function is flatter.

A. General Results

The Fisher information matrix is given by (30).

$$\mathcal{I}(\mathbf{x}) = \begin{bmatrix} \mathcal{I}(\mathbf{p}) & \cdot \\ \cdot & \mathcal{I}(\mathbf{v}) \end{bmatrix} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \begin{bmatrix} \mathcal{I}_i(\mathbf{p}) & \cdot \\ \cdot & \mathcal{I}_i(\mathbf{v}) \end{bmatrix} \tag{30}$$

We use $\mathcal{I}(\mathbf{x})$, $\mathcal{I}(\mathbf{p})$ and $\mathcal{I}(\mathbf{v})$ to denote the Fisher information matrix defined by considering only the parameters $\mathbf{x}$, $\mathbf{p}$ and $\mathbf{v}$ respectively. Both $\mathcal{I}(\mathbf{p})$ and $\mathcal{I}(\mathbf{v})$ turn out to be principal submatrices of $\mathcal{I}(\mathbf{x})$. In all cases, independent measurements from additional sensors in general positions will never decrease the total information in each $\mathbf{x}$, $\mathbf{p}$ and $\mathbf{v}$.

Proposition 6. *The condition $n \geq 4$ is a necessary condition for $\mathcal{I}(\mathbf{x})$ to be non-singular.*

Proof: Recall that $\mathcal{I}(\mathbf{x}) = \nabla F^\top \mathbf{R}_{\mathbf{f}}^{-1} \nabla F$ or given $\mathbf{R}_{\mathbf{f}} = \mathrm{diag}\left(\sigma_1^2, \ldots, \sigma_n^2\right)$ we have [20]

$$\mathcal{I}(\mathbf{x}) = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \nabla f_i^\top \nabla f_i \tag{31}$$

which is a sum of matrices each with rank at most 1. Now a well-known result states that a rank-k matrix can be written as the sum of k rank-1 matrices but not fewer. This immediately implies our result and completes the proof. ■

Proposition 7. *The following statements concerning efficient estimators hold.*

1) *If n is finite then no efficient estimator exists for $\mathbf{x}$.*
2) *If $\mathbf{p}$ is known and n is finite then an efficient estimator for $\mathbf{v}$ exists and is given by the standard linear maximum likelihood estimator.*
3) *If $\mathbf{v}$ is known and n is finite then no efficient estimator exists for $\mathbf{p}$.*

Proof: This result follows from a general result concerning efficient estimators given in Theorem 1 and 2 in [18]. ■

We do not consider the design of unbiased (but inefficient) estimators for either $\mathbf{p}$ or $\mathbf{x}$ in this work. However, we know that no unbiased estimator for $\mathbf{x}$ exists with a finite variance when $n < 4$. Similarly, no unbiased estimator for $\mathbf{p}$ or $\mathbf{v}$ exists with a finite variance when $n < 2$.

B. Discussion on the Cramer-Rao Bound

The Cramer-Rao inequality assumes an unbiased estimation algorithm and an estimator which achieves the inequality is called an efficient estimator. An efficient estimator does not exist for $\mathbf{x}$ or $\mathbf{p}$ when n is finite (For example, the well-known maximum likelihood localization techniques are only unbiased and efficient when the number of sensors approaches infinity.) but does exist for $\mathbf{v}$ when $\mathbf{p}$ is given. Even if an efficient estimator does not exist then it may be possible to design an unbiased estimator. This possibility is not explored in this work. In practice, a system designer may be constrained in their choice of parameter estimator. Likely, the estimation technique used in practice will be biased [21], [22]. The Cramer-Rao bound for unbiased estimators is still an interesting benchmark with which intuitively pleasing results and performance measures can be derived. However, these results can only be considered as a guide.

It is interesting that the variance (or mean-square-error) of an estimate can sometimes be made smaller at the expense of increasing the bias [23]. The work of [24], [25] explores the concept of bias-variance trade offs in estimation. In [17], [25] a *biased* Cramer-Rao inequality and in [24], [25] a *uniform* Cramer-Rao inequality are developed and can be used to study this so-called bias-variance trade off. These ideas are yet to be fully explored in the localization and target tracking literature.

VI. On the Fisher Information for Velocity Estimation

Doppler-based measurements are often used to estimate the target velocity alone. In this section we explore the relationship between the transmitter and sensor positions and the velocity estimation error lower bound defined by $\mathcal{I}_i(\mathbf{v})$. To this end, consider the sub-matrix

$$\mathcal{I}(\mathbf{v}) = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \begin{bmatrix} \cos^2(\phi_i) & \frac{\sin(2\phi_i)}{2} \\ \frac{\sin(2\phi_i)}{2} & \sin^2(\phi_i) \end{bmatrix} \tag{32}$$

where $\cos\phi_i = \psi_{i,1}$, and $\sin\phi_i = \psi_{i,2}$.

Firstly, we note that optimal sensor placement for velocity estimation is equivalent to the optimal sensor placement for range-based localization as outlined in [26]. However, importantly, the optimal sensor placement for estimating the target velocity does not depend on the velocity itself. Hence, such placements can be used in practice to estimate the velocity of the target when other measures are available for positioning; e.g. we have already discussed a number of hybrid scenarios in which the additional measurements provide access to positioning information as opposed to velocity.

Moreover, with regards to the optimal sensor placement for velocity estimation we assume the error variance in doppler measurements is independent of their true value. While this is not unreasonable in many applications, this assumption can be relaxed such that the variance is different across sensors. Moreover, we can even consider cases in which the standard deviation is multiplied by a percentage of the true range value between the target and individual sensors. In both these cases the Fisher information matrix for velocity estimation becomes very similar to the Fisher information matrix for bearing-only localization and we point to [26] for the details.

We summarize the result on the optimal sensor placement for estimating the velocity of the target in the following proposition.

Proposition 8. *Let the angle subtended at the target by two sensors i and j be denoted by $\vartheta_{ij} = \vartheta_{ji}$. One set of optimal sensor placement is characterized by*

$$\vartheta_{ij} = \vartheta_{ji} = \frac{2}{n}\pi \tag{33}$$

for all adjacent sensor pairs $i, j \in \{1, \ldots, n \geq 4\}$ with $|j-i| = 1$ or $|j-i| = n-1$, and then by a possible application of the following actions on (33):

1) *Changing the true individual sensor-target ranges, i.e. moving a sensor from $\mathbf{s}_i$ to $\mathbf{p}+k(\mathbf{s}_i-\mathbf{p})$ for some $k > 0$.*
2) *Reflecting a sensor about the emitter position, i.e. moving a sensor from $\mathbf{s}_i$ to $2\mathbf{p}-\mathbf{s}_i$.*

For example, Fig. 2 illustrates two optimal sensor placement scenarios obtained from each other by reflecting a particular sensor about the emitter position.

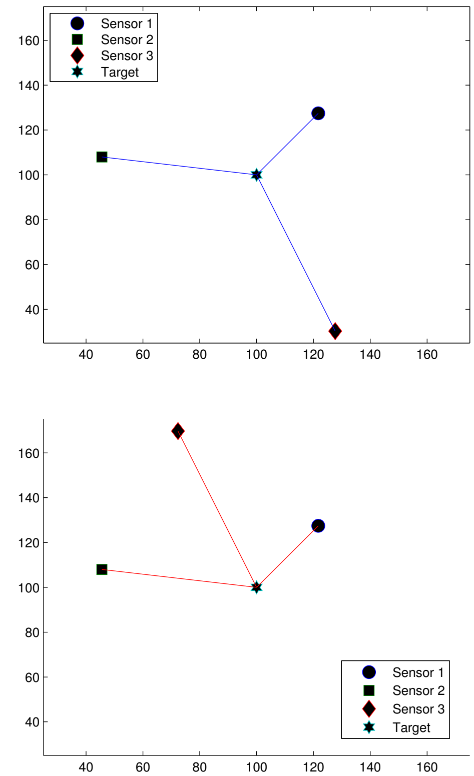

Fig. 2. This figure illustrates two scenarios obtained from each other by reflecting a particular sensor about the emitter position. This reflection does not affect the optimality of the sensor-target configuration.

For more information on the proof of Proposition 8 and further details the reader may refer to [26].

VII. Illustrative Example

In this section we demonstrate the performance of the algorithm introduced in Section IV to estimate the position and the velocity of the target. We consider the setting depicted in Fig, 3. We consider the case where the measurements are corrupted by ten different levels of noise, where the noise is assumed to be Gaussian with zero mean and variable variance (and independent at each sensor). The carrier frequency is assumed to be constant for all sensors and equal to 150 MHz. Moreover, the nominal doppler sifts are: $\delta_1 = -9.2664$, $\delta_2 = 3.1640$, $\delta_3 = 3.1640$, $\delta_4 = -7.8162$, $\delta_5 = -9.4941$, $\delta_6 = 7.3782$, and $\delta_7 = 9.1797$ all in Hz. The error in the estimates of position and the velocity for different noise levels after repeating the scenario for one hundred times when six and seven measurements are used are depicted Fig. 4 respectively (In the six measurement case, it is s_7 measurements which are omitted.). Furthermore, we employ a maximum likelihood estimator that is initialized with the solution of the optimization problem (25) to estimate the position and the velocity of the target. The obtained maximum liklihood estimates are presented in Fig. 4. Note, that in the simulations considered, when the estimator is initialized at a random value the estimate does not converge to a value close to the solution for any of the considered noise levels. One notices that using seven measurements the maximum liklihood solution when initalized at the solution of optimization problem (25) outperforms the Cramer-Rao lower bound. We conjecture that it is due to the possibility of obtaining a biased estimate using the methods outlined in this paper.

Additionally, note that the setting presented in Fig. 3 shows elements of a degenerate geometry as well: sensors at positions s_2 and s_3, and the target are collinear. However, here due to the presence of other sensors at non-collinear positions the estimation can be carried out effectively. Due to this issue, if one considers only five measurements taken by $s_1, \ldots, s_5$ the localization problem cannot be solved.

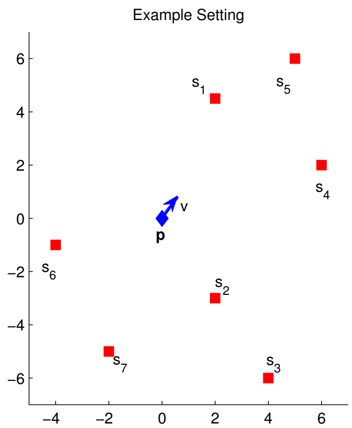

Fig. 3. The setting considered in the illustrative example. The squares denote the position of the sensors and the diamond is the target.

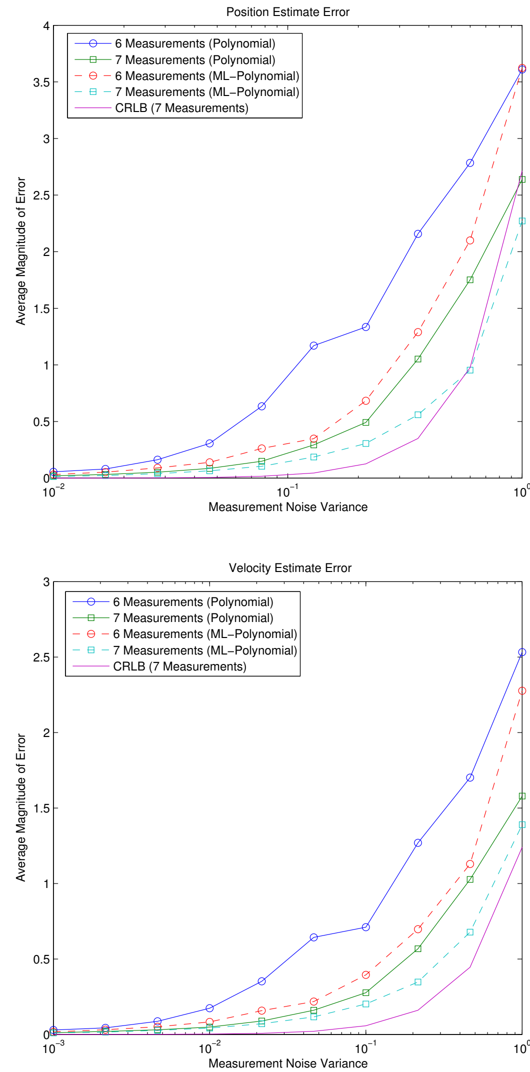

Fig. 4. The error in the estimate of the position and the velocity of the target using the proposed polynomial optimization method (Polynomial) and maximum likelihood initialized at the solution of the optimization problem (ML-Polynomial) after repeating the scenario 100 times with variable noise variances when six and seven sensors are used.

VIII. Concluding Remarks and Future Directions

The minimum number of doppler shift measurements necessary to have a finite number of solutions for the unknown target position and velocity is calculated analytically via algebraic arguments. Additionally, we stated the necessary and sufficient number of generic measurements to have a unique solution for the target parameters. Later, the same problem has been studied where in addition to doppler shift measurements, other types of measurements are available, e.g. bearing or distance to the target from each of the sensors. Finally, a method based on polynomial optimization is introduced to calculate an estimate for the position and the velocity of the target using noisy doppler shift measurements. A numerical example is presented to demonstrate the performance of this algorithm.

Moreover, some remarks concerning the Cramer-Rao inequality and its relationship to the estimation problem were given. For the case of doppler-based target velocity estimation we completely characterized the sensor-target geometry and provided a number of conditions on the optimal placement of the sensors and the transmitters.

A possible future research direction is to design a dynamical estimator to estimate the position and the velocity of the target measuring the doppler shift measurements continuously. Alternatively, one could consider the notion of constraint-based optimization for localization as discussed in, e.g., [19], [27]–[31].

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